# Uniform sup-norm bounds for Siegel cusp forms

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# Structure of the talk

# Motivation

## Sup-norm bounds on the upper half-plane

# Sup-norm bounds on $\mathbb{H}$



- $\mathbb{H} := \{z = x + iy \mid y > 0\}$, upper half-plane
- $\Gamma \subsetneq \mathrm{SL}(2, \mathbb{R})$, Fuchsian subgroup of the first kind
- $\mathcal{S}_k(\Gamma)$: space of cusp forms on $\mathbb{H}$ of weight $k$ w.r.t $\Gamma$
- $\{f_j\}_{1 \leq j \leq d}$ O.N.B. on $\mathcal{S}_k(\Gamma)$ w.r.t. Petersson inner product.

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**Theorem (Friedman, Jorgenson & Kramer, 2016)**

$S_k^\Gamma(z) := \sum_{j=1}^{d} y^k |f_j(z)|^2 \quad (z \in \mathbb{H}, k \geq 2)$

$$\sup_{z \in \mathbb{H}} S_k^\Gamma(z) \leq \begin{cases} c_\Gamma\, k & (\Gamma \text{ cocompact}), \\ c_\Gamma\, k^{3/2} & (\Gamma \text{ cofinite}), \end{cases}$$

*where $c_\Gamma > 0$ is a positive real number depending only on $\Gamma$.*

*Furthermore, this bound is uniform in the sense that if we fix a group $\Gamma_0 \subsetneq \mathrm{SL}(2, \mathbb{R})$ and take $\Gamma$ to be a subgroup of $\Gamma_0$ of finite index, then $c_\Gamma$ depends only on the fixed group $\Gamma_0$.*

# Results

## Sup-norm bounds on the Siegel upper half-space

# Generalization

- $\mathbb{H}_n = \{Z = X + iY \mid X, Y \in \mathbb{R}^{n\times n}, X = X^t, Y = Y^t, Y > 0\}$ Siegel upper half-space of degree $n$
- $\mathrm{Sp}(n,\mathbb{R}) := \{g \in \mathbb{R}^{2n\times 2n} \mid g^t J_n g = J_n\}$ with $J_n := \left(\begin{smallmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{smallmatrix}\right)$, real symplectic group of degree $n$
- $Z \mapsto gZ = (AZ+B)(CZ+D)^{-1}$ $\left(g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(n,\mathbb{R})\right)$
- $\Gamma \subsetneq \mathrm{Sp}(n,\mathbb{R})$ arithmetic subgroup, e.g., $\Gamma_n := \mathrm{Sp}(n,\mathbb{Z})$

extra

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- $\Gamma \subsetneq \mathrm{Sp}(n, \mathbb{R})$ arithmetic subgroup, e.g., $\Gamma_n := \mathrm{Sp}(n, \mathbb{Z})$
- $\mathcal{S}_k^n(\Gamma)$: space of cusp forms on $\mathbb{H}_n$ of weight $k$ w.r.t $\Gamma$
- $\{f_j\}_{1 \leq j \leq d}$, a basis of $\mathcal{S}_k^n(\Gamma)$ orthonormal with respect to the Petersson inner product on $\mathcal{S}_k^n(\Gamma)$.
- $S_k^\Gamma(Z) := \sum_{j=1}^{d} \det(Y)^k |f_j(Z)|^2 \quad (Z \in \mathbb{H}_n)$

extra

# Sup-norm bounds on $\mathbb{H}_n$

## Theorem

- $\Gamma \subsetneq \mathrm{Sp}(n, \mathbb{R})$ *arithmetic subgroup*
- $k \geq n+1$

*Then, for all* $n \geq 2$, *we have*

$$\sup_{Z \in \mathbb{H}_n} S_k^{\Gamma}(Z) \leq \begin{cases} c_{n,\Gamma}\, k^{n(n+1)/2} & (\Gamma \text{ cocompact}), \\ c_{n,\Gamma}\, k^{3n(n+1)/4} & (\Gamma \text{ cofinite}), \end{cases}$$

*where* $c_{n,\Gamma} > 0$ *is a positive real number depending only on the degree* $n$ *of* $\mathbb{H}_n$ *and the group* $\Gamma$.

*Furthermore, this bound is uniform in the sense that if we fix a group* $\Gamma_0 \subsetneq \mathrm{Sp}(n, \mathbb{R})$ *and take* $\Gamma$ *to be a subgroup of* $\Gamma_0$ *of finite index, then the constant* $c_{n,\Gamma}$ *in these bounds depends only on the degree* $n$ *and the fixed group* $\Gamma_0$.

# Strategy of proof

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- $\mathcal{V}_k^n(\Gamma)$: the space of real analytic functions $\varphi\colon \mathbb{H}_n \to \mathbb{C}$ with the transformation behaviour

$$\varphi(\gamma Z) = \left(\frac{\det(CZ + D)}{\det(C\overline{Z} + D)}\right)^{k/2} \varphi(Z) \quad \left(\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma\right)$$

- Petersson inner product and norm defined on $\mathcal{V}_k^n(\Gamma)$.
- $\mathcal{H}_k^n(\Gamma) := \{\varphi \in \mathcal{V}_k^n(\Gamma) \mid \|\varphi\| < \infty\}$, the Hilbert space of square integrable functions in $\mathcal{V}_k^n(\Gamma)$.

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- $\Delta$: Laplace–Beltrami operator on $\mathbb{H}_n$
- Siegel–Maaß Laplacian of weight $k$: $\Delta_k = \Delta - \mathrm{tr}\left(ikY \dfrac{\partial}{\partial X}\right)$

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- $\Delta$: Laplace–Beltrami operator on $\mathbb{H}_n$
- Siegel–Maaß Laplacian of weight $k$: $\Delta_k = \Delta - \mathrm{tr}\left(ikY\frac{\partial}{\partial X}\right)$
- $\Delta_k$ extends to an essentially self-adjoint linear operator acting on a dense subspace of $\mathcal{H}_k^n(\Gamma)$.

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- Laplace eq. $(\Delta_k + \lambda)\varphi = 0$ satisfy $\lambda \geq \frac{nk}{4}((n+1) - k)$

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**Connecting Siegel cusp forms to $\Delta_k$**

$\mathcal{S}_k(\Gamma) \cong \ker(\Delta_k + \frac{nk}{4}((n+1)-k))$ induced by $f \mapsto \det(Y)^{k/2} f$

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**Connecting heat kernel to $S_k^\Gamma(Z)$**

$$\lim_{t\to\infty} \exp\left(\frac{nk}{4}((n+1)-k)\,t\right) K_t^{(k,\Gamma)}(Z) = \sum_{j=1}^{d} \det(Y)^k |f_j(Z)|^2$$

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**Connecting heat kernel to $S_k^\Gamma(Z)$**

$$\lim_{t\to\infty} \exp\left(-\frac{nk}{4}(k-(n+1))\,t\right) K_t^{(k,\Gamma)}(Z) = S_k^\Gamma(Z) \quad (k \geq n+1)$$

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**Connecting heat kernel to $S_k^\Gamma(Z)$**

$$\exp\left(-\frac{nk}{4}(k-(n+1))\,t\right) K_t^{(k,\Gamma)}(Z) \geq S_k^\Gamma(Z) \quad (t > 0)$$

# Heat kernel

# Heat kernel on Siegel upper half-space



The heat kernel on $\mathbb{H}_n$ corresponding to the Laplace–Beltrami operator $\Delta = \Delta_0$ is obtained as:

**Heat kernel on $\mathbb{H}_n$**

$$K_t(2R) = c_n \frac{\exp\left(-\sum_{j=1}^{n} j^2 t/4\right)}{t^{n^2+n/2}} \int\limits_{q\in K} \frac{\varepsilon(\varrho(r,q)) \exp\left(-\sum_{j=1}^{n} \varrho_j(r,q)^2/t\right)}{\delta(\varrho(r,q))} \, \mathrm{d}\mu(q)$$

where,

$R = R(Z,W)\,(Z,W \in \mathbb{H}_n)$ is a $(n \times n)$ diagonal matrix coming from the eigenvalues of the cross-ratio matrix of $Z$ and $W$.

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where,

$$R = \begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_n \end{pmatrix} \quad r = \begin{pmatrix} R & 0 \\ 0 & -R \end{pmatrix} \quad (r_j \in \mathbb{R}_{\geq 0})$$

$$P = \begin{pmatrix} \varrho_1 & & 0 \\ & \ddots & \\ 0 & & \varrho_n \end{pmatrix} \quad \varrho = \begin{pmatrix} P & 0 \\ 0 & -P \end{pmatrix} \quad (\varrho_j \in \mathbb{R})$$

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where,

- $qe^r\overline{q}^t = ue^{\varrho}\overline{u}^t \in \mathrm{Sp}(n,\mathbb{C})$, Hermitian.
- $r$ and $\varrho$ symplectic diagonal.
- $q \in K = \mathrm{Sp}(n,\mathbb{C}) \cap O(2n,\mathbb{C})$
- $u \in U = \mathrm{Sp}(n,\mathbb{C}) \cap U(2n)$
- Hard to explicitly calculate $\varrho$ in terms of $r$ and $q$.

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where,

$$\varepsilon(\varrho) = \prod_{1 \le j \le n} \varrho_j \prod_{1 \le j < k \le n} (\varrho_j + \varrho_k) \prod_{1 \le j < k \le n} (\varrho_j - \varrho_k)$$

$$\delta(\varrho) = \prod_{1 \le j \le n} \mathrm{sh}(\varrho_j) \prod_{1 \le j < k \le n} \mathrm{sh}\left(\frac{\varrho_j + \varrho_k}{2}\right) \prod_{1 \le j < k \le n} \mathrm{sh}\left(\frac{\varrho_j - \varrho_k}{2}\right)$$

# Heat kernel of weight k

The heat kernel $K_t^{(k)}(2R)$ of weight $k$ on $\mathbb{H}_n$ is immediately obtained from the previous formula by inserting the factor $\det(h(q))^{2k}$, i.e.,

$$K_t^{(k)}(2R) = c_n \frac{e^{-\sum_{j=1}^{n} j^2 t/4}}{t^{n^2+n/2}} \int_K \cdots \det(h(q))^{2k} \, \mathrm{d}\mu(q),$$

where the matrix $h(q) \in \mathbb{C}^{n \times n}$ is obtained as follows:

- Write $q \in K$ as $q = q_0 q_h$ with $q_0$ real orthogonal and
$$q_h = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$
is hermitian orthogonal.
- Then, we obatin $h(q) = A + iB$, which is hermitian.

# Heat kernel of weight k



From the parametrization
$u = \begin{pmatrix} V & 0 \\ 0 & \overline{V} \end{pmatrix} \begin{pmatrix} \cos(\Theta) & \sin(\Theta) \\ -\sin(\Theta) & \cos(\Theta) \end{pmatrix} \begin{pmatrix} W & 0 \\ 0 & \overline{W} \end{pmatrix}$ of $U$ and the relation $qe^r\overline{q}^t = ue^\varrho\overline{u}^t$, one obtains

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$$\det(h(q)) = \frac{\det(\cos(\Theta) W e^P \overline{W}^t \cos(\Theta) + \sin(\Theta) \overline{W} e^{-P} W^t \sin(\Theta))}{\prod_{j=1}^n \mathrm{ch}^k(r_j)}$$

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$$\det(h(q)) \leq \exp\Big(\sum_{j=1}^{n} |\varrho_j|\Big) / \prod_{j=1}^{n} \mathrm{ch}^k(r_j)$$

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$$\det(h(q)) \leq \exp\left(\sum_{j=1}^{n} |\varrho_j|\right) / \prod_{j=1}^{n} \mathrm{ch}^k(r_j)$$

**Periodized weight-$k$ heat kernel on $\Gamma\backslash\mathbb{H}_n$**

$$K_t^{(k,\Gamma)}(Z) := \sum_{\gamma\in\Gamma} \det\left(\frac{Z-\gamma\overline{Z}}{\gamma Z-\overline{Z}}\right)^{k/2} \det\left(\frac{C\overline{Z}+D}{CZ+D}\right)^{k/2} K_t^{(k)}(2R(Z,\gamma Z))$$

# Concluding the proof

# Compact case



- Let $M$ be compact. By a counting function argument

$$\sum_{\gamma\in\Gamma} K_t^{(k)}(R(Z,\gamma Z)) \leq c_{n,\Gamma} \int_{(r_j)=0} K_t^{(k)}(2R)\,|\delta(2r)| \bigwedge_{j=1}^{n} \mathrm{d}r_j$$

$$= c_{n,\Gamma} \int_{(r_j)=0}^{\infty} \int_{q\in K} \cdots \quad \mathrm{d}\mu(q) \wedge \bigwedge_{j=1}^{n} \mathrm{d}r_j$$

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$$= c_{n,\Gamma} \int\limits_{(r_j)=0}^{\infty} \int\limits_{q\in K} \cdots \quad \mathrm{d}\mu(q) \wedge \bigwedge_{j=1}^{n} \mathrm{d}r_j$$

- From $qe^r\overline{q}^t = ue^{\varrho}\overline{u}^t$, using change of variables $|\delta(2r)| \bigwedge_{j=1}^{n} \mathrm{d}r_j \wedge \mathrm{d}\mu(q) = c_n\,\delta(\varrho)^2 \bigwedge_{j=1}^{n} \mathrm{d}\varrho_j \wedge \mathrm{d}\mu(u)$, the right hand integral becomes

$$\sum_{\gamma\in\Gamma} K_t^{(k)}(R(Z,\gamma Z)) \leq c_{n,\Gamma} \int\limits_{(\varrho_j)=-\infty}^{\infty} \int\limits_{u\in U} \cdots \quad \mathrm{d}\mu(u) \wedge \bigwedge_{j=1}^{n} \mathrm{d}\varrho_j$$

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- Taking the highest values of $k$ and $t$ we have $S_k^{\Gamma}(Z) \leq c_{n,\Gamma}\, k^{n(n+1)}\, t^{\frac{n(n+1)}{2}} \mu(\sqrt{t})$.

# Compact case

- This gets rid of the semi-explicit nature of the integral and can be explicitly bounded by a series of Gamma integrals which can be easily evaluated to a polynomial in $k$ and $t$.
- Taking the highest values of $k$ and $t$ we have $S_k^{\Gamma}(Z) \leq c_{n,\Gamma}\, k^{n(n+1)}\, t^{\frac{n(n+1)}{2}} \mu(\sqrt{t})$.
- Now multiplying both sides of the above inequality by $e^{-kt}$ and integrating over $t \in [0, \infty]$, we have

$$S_k^{\Gamma}(Z) \leq c_{n,\Gamma}\, k^{n(n+1)/2} \qquad (Z \in \mathbb{H}_n),$$

which is the requisite compact bound.

# Non-compact case

- With some limiting argument on the heat kernel:

$$\sup_{Z \in \mathbb{H}_n} S_k^{\Gamma}(Z) \leq c_n \, k^{n(n+1)/2} \sum_{\gamma \in \Gamma} \frac{1}{\prod_{j=1}^{n} \mathrm{ch}^k(r_j(Z, \gamma Z))}$$

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- Using the commensurability of $\Gamma$ with $\Gamma_n$, we shift to the standard picture for 'cusps at infinity' for $\Gamma_n$.

Sup-norm bounds for Siegel cusp forms



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- Then with a maximum-modulus argument, we show that in suitably chosen cusp-neighbourhoods, the compact bound holds.

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- Using the commensurability of $\Gamma$ with $\Gamma_n$, we shift to the standard picture for 'cusps at infinity' for $\Gamma_n$.
- Then with a maximum-modulus argument, we show that in suitably chosen cusp-neighbourhoods, the compact bound holds.
- Too far away from the cusps, obviously the compact bound holds

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- With some limiting argument on the heat kernel:

$$\sup_{Z \in \mathbb{H}_n} S_k^{\Gamma}(Z) \leq c_n \, k^{n(n+1)/2} \sum_{\gamma \in \Gamma} \frac{1}{\prod_{j=1}^{n} \operatorname{ch}^k(r_j(Z, \gamma Z))}$$

- Using the commensurability of $\Gamma$ with $\Gamma_n$, we shift to the standard picture for 'cusps at infinity' for $\Gamma_n$.
- Then with a maximum-modulus argument, we show that in suitably chosen cusp-neighbourhoods, the compact bound holds.
- Too far away from the cusps, obviously the compact bound holds
- Thus, left to determine the bound only in the annulus: $\{Z = X + iY \in \mathscr{F}_n \mid \varepsilon < \lambda_n(Y) \leq \frac{k}{2c_2(n)}\}$ $\subsetneq \{Z = X + iY \in \mathscr{F}_n \mid Y \leq \frac{k}{2c_2(n)} \mathbb{1}_n\}$

# Non-compact case

- In this region, we split the sum into $\sum_{\gamma\in\Gamma\setminus\Gamma_\infty}+\sum_{\gamma\in\Gamma_\infty}$.

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## Standard parabolic matrices

$$\Gamma_\infty=\left\{\left(\begin{smallmatrix}A & AS\\ 0 & A^{-t}\end{smallmatrix}\right)\,\middle|\, A=\left(\begin{smallmatrix}\mathbb{1}_j & 0\\ L & \mathbb{1}_{n-j}\end{smallmatrix}\right),\ S=\left(\begin{smallmatrix}0 & H^t\\ H & S_2\end{smallmatrix}\right),\ 1\leq j\leq n-1\right\},$$

where $L,H\in\mathbb{Z}^{(n-j)\times j}$ and $S_2\in\mathbb{Z}^{(n-j)\times(n-j)}$, $S_2=S_2^t$.

Sup-norm bounds for Siegel cusp forms

Motivation

Results

Strategy of proof

Heat kernel

Concluding the proof

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- The sum $\sum_{\gamma\in\Gamma\backslash\Gamma_\infty}$ gives only compact bound.
- The largest contribution in the sum $\sum_{\gamma\in\Gamma_\infty}$ comes from

$$\Gamma_\infty^0 = \left\{ \left( \begin{smallmatrix} \mathbb{1}_n & S \\ 0 & \mathbb{1}_n \end{smallmatrix} \right) \,\middle|\, S = S^t \in \mathbb{Z}^{n\times n} \right\}.$$

- $\left( \begin{smallmatrix} \mathbb{1}_n & S \\ 0 & \mathbb{1}_n \end{smallmatrix} \right) Z = Z + S = (X+S) + iY \qquad (Z = X + iY \in \mathbb{H}_n).$

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We estimate this contribution by

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This gives the requisite non-compact bound.

# Thank you!

# Extra definitions



## Definition (Siegel cusp form)

Let $\Gamma \subset \mathrm{Sp}_n(\mathbb{R})$ be a subgroup commensurable with $\mathrm{Sp}_n(\mathbb{Z})$, i.e., the intersection $\Gamma \cap \mathrm{Sp}_n(\mathbb{Z})$ is a finite index subgroup of $\Gamma$ as well as of $\mathrm{Sp}_n(\mathbb{Z})$.

We let $\gamma_j \in \mathrm{Sp}_n(\mathbb{Z})$ ($j = 1, \ldots, h$) denote a set of representatives for the left cosets of $\Gamma \cap \mathrm{Sp}_n(\mathbb{Z})$ in $\mathrm{Sp}_n(\mathbb{Z})$. Then, a *Siegel cusp form of weight $k$ and degree $n$ for $\Gamma$* is a function $f\colon \mathbb{H}_n \longrightarrow \mathbb{C}$ satisfying the following conditions:

(i) $f$ is holomorphic;

(ii) $f(\gamma Z) = \det(CZ + D)^k f(Z)$ for all $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$;

(iii) given $Y_0 \in \mathrm{Sym}_n(\mathbb{R})$ with $Y_0 \gg 0$, the quantities $\det(C_j Z + D_j)^{-k} f(\gamma_j Z)$ become arbitrarily small in the region $\{Z = X + iY \in \mathbb{H}_n \mid Y \geq Y_0\}$ for the set of representatives $\gamma_j = \left(\begin{smallmatrix} A_j & B_j \\ C_j & D_j \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{Z})$. back

# Extra definitions

- Distance matrix $R(Z,W)$ on $\mathbb{H}_n$ is given by

$$R(Z,W)=\begin{pmatrix} r_1(Z,W) & & 0 \\ & \ddots & \\ 0 & & r_n(Z,W) \end{pmatrix} \quad (Z,W\in\mathbb{H}_n)$$

$r_j(Z,W)$ related to the eigenvalues $\rho_j(Z,W)$ of the cross-ratio matrix

$$\rho(Z,W)=(Z-W)(\overline{Z}-W)^{-1}(\overline{Z}-\overline{W})(Z-\overline{W})^{-1}$$

by the relation

$$\exp(2r_j(Z,W))=\frac{1+\sqrt{\rho_j(Z,W)}}{1-\sqrt{\rho_j(Z,W)}} \qquad (1\leq j\leq n).$$

- Siegel metric on $\mathbb{H}_n$ given by:

$$d\mu_n(Z)=\frac{\bigwedge_{1\leq j\leq k\leq n} dx_{j,k}\wedge dy_{j,k}}{\det(Y)^{n+1}} \qquad (z_{j,k}=x_{j,k}+iy_{j,k})$$